Junior problems

J289. Let a be a real number such that $0 \le a < 1$. Prove that

$$\left\lfloor a\left(1+\left\lfloor \frac{1}{1-a}\right\rfloor\right)\right\rfloor + 1 = \left\lfloor \frac{1}{1-a}\right\rfloor.$$

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Since $0 \le a < 1$, then $0 < 1-a \le 1$.
If $k \in \mathbb{N}$ such that $\frac{1}{1+k} < 1-a \le \frac{1}{k}$, then $k \le \frac{1}{1-a} < k+1$, and $\frac{k-1}{k} \le a < \frac{k}{k+1}$, so $\left\lfloor \frac{1}{1-a}\right\rfloor = k$.
On the other hand, for the left-hand side of the proposed identity we have

$$\begin{aligned}\left\lfloor a\left(1+\left\lfloor \frac{1}{1-a}\right\rfloor\right)\right\rfloor + 1 &= \lfloor a(1+k)\rfloor + 1\\ &= k-1+1 = k.\end{aligned}$$

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